

Transcendental ℓ -adic Galois representations

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1 Introduction

Let F be a number field and $G_F = \text{Gal}(\overline{F}/F)$ be its absolute Galois group. Let \mathbb{C}_ℓ be the completion of $\overline{\mathbb{Q}_\ell}$. In this paper we study continuous, *transcendental* ℓ -adic Galois representations $\rho : G_F \rightarrow \text{GL}_n(\mathbb{C}_\ell)$. Such representations which arise classically (for example from geometry) have models over a finite extension of \mathbb{Q}_ℓ . By an argument that is part of the “folklore”, and that we have seen attributed to Florian Pop, one does not get any “new representations” if one considers representations $\rho : G_F \rightarrow \text{GL}_n(\overline{\mathbb{Q}_\ell})$ as these have models over some K that is a finite extension of \mathbb{Q}_ℓ . This follows simply from the fact that a compact subgroup C of $\text{GL}_n(\overline{\mathbb{Q}_\ell})$ in fact lies in $\text{GL}_n(K)$ for K a sufficiently large finite extension of \mathbb{Q}_ℓ (*Baire category theorem*: namely by this we know that for some finite K'/\mathbb{Q}_ℓ the intersection of C and $\text{GL}_n(K')$ is an open, and hence finite index, subgroup of C).

We describe below (see Theorem 3.10) the construction of an example of a transcendental representation, which

- is semisimple, and therefore unramified at a density one set of places by a result we prove below (see Theorem 2.5),
- at a density one set of unramified places has characteristic polynomials of Frobenii defined over $\overline{\mathbb{Q}_\ell}$,

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- does not have a model over $\overline{\mathbb{Q}_\ell}$, or equivalently over any finite extension of \mathbb{Q}_ℓ .

By another result we prove below (see Corollary 4.4), such representations are necessarily *infinitely ramified*. In the course of this paragraph, we have also given a description of the three main results of this work.

The study of \mathbb{C}_ℓ -semilinear representations of $\text{Gal}(\overline{\mathbb{Q}_\ell}/\mathbb{Q}_\ell)$ has been a central object of study in the subject of p -adic (or ℓ -adic in present notation!) Hodge theory for more than 30 years. The \mathbb{C}_ℓ -linear representations that we study here of global Galois groups have to our knowledge not been studied. The *rationale* that we can offer for the representations considered here, besides the obvious one that we find them diverting, may be summarised as follows.

1. Transcendental ℓ -adic representations include the study of big Galois representations, that have been studied earlier in the work of Hida, Mazur et al, as one has continuous embeddings $\mathbb{Z}_\ell[[X_1, \dots, X_r]] \hookrightarrow \mathbb{C}_\ell$ (for arbitrary r).
2. As the theory of complex representations of various kinds of groups illustrates, it is natural to study representations over complete, algebraically closed fields that are further minimal. Here by minimal we mean in the sense of \mathbb{C} being the complete, algebraic closure of \mathbb{R} , the ∞ -adic completion of its prime field \mathbb{Q} , or \mathbb{C}_ℓ being the completion of the algebraic closure of \mathbb{Q}_ℓ , the ℓ -adic completion of its prime field \mathbb{Q} .
3. This is a continuation of the point above. Galois representations with values in $\text{GL}_n(\mathbb{C}_\ell)$ can arise as limits of representations that are all defined over $\overline{\mathbb{Q}_\ell}$.

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2 Cebotarev density and ramification

In this section, we consider continuous Galois representations taking values in the field of fractions K of a valuation ring V . We do not assume that V is discrete or complete or that its residue field is finite. Our goal is to

prove that continuous semisimple Galois representations taking values in K are unramified outside a thin set of primes. This was shown in the locally compact case in [Kh-Raj]. We follow the basic strategy of [Kh-Raj]. A crucial part of this paper, the estimate of volumes of “tubular neighborhoods” of subvarieties (following J-P. Serre [S2]), needs to be replaced by a softer technique. Our method does not give the Hausdorff dimension as in the paper of Serre, but it does allow us to prove that the measure of the subset of a compact subgroup Γ of $\mathrm{GL}_n(K)$ lying in a sufficiently small tubular neighborhood of a codimension ≥ 1 subvariety of the Zariski closure of Γ can be made as small as desired. This also proves a Cebotarev density theorem for transcendental ℓ -adic representations (see Theorem 2.4) that might be of independent interest.

For the convenience of the reader, we begin by recalling that the standard argument for finding an integral basis for representations of compact groups does not depend on the compactness of V .

Lemma 2.1 *Every finitely generated torsion-free module over a valuation ring is free.*

Proof. Let V be a valuation ring with valuation v and M a finitely generated torsion-free V -module. Let m_1, m_2, \dots, m_k be a minimal set of generators of M . We claim they form a basis. If not, after reindexing there exists a linear combination $a_1 m_1 + \dots + a_r m_r = 0$, where $1 \leq r \leq k$, all a_i are non-zero, and $v(a_1) \geq v(a_2) \geq \dots \geq v(a_r)$. Then, as M is torsion-free,

$$m_r = -\frac{a_1}{a_r} m_1 - \dots - \frac{a_{r-1}}{a_r} m_{r-1} \in V m_1 + \dots + V m_{r-1},$$

contrary to the minimality of $\{m_i\}$. \square

Lemma 2.2 *Let V be a valuation ring with fraction field K , n a positive integer, and Γ a compact subgroup of $\mathrm{GL}_n(K)$. Then Γ can be conjugated within $\mathrm{GL}_n(K)$ into a subgroup of $\mathrm{GL}_n(V)$.*

Proof. Let $\Gamma^\circ = \Gamma \cap \mathrm{GL}_n(V)$. Then Γ° is an open subgroup of Γ and therefore of finite index. Consider the coset decomposition $\Gamma = \gamma_1 \Gamma^\circ \cup \dots \cup \gamma_m \Gamma^\circ$. Let $M = \gamma_1 V^n + \dots + \gamma_m V^n$. Then M is a finitely generated V -submodule of K^n (which is torsion-free). Therefore it is torsion-free, and as it spans K^n , isomorphic to V^n . It follows that there exists an element of $\mathrm{GL}_n(K)$ mapping the original basis of V^n to a basis of M . \square

2.1 Cebotarev density

Our goal is to show that subvarieties of positive codimension in the Zariski closure of the image of a Galois representation capture only a density-zero set of Frobenius elements. We do this by considering “tubular neighborhoods” of such subvarieties and showing that as “radius” goes to 0, measure goes to 0 as well. The following proposition is the key.

Proposition 2.3 *Let Γ denote a compact subgroup of $\mathrm{GL}_n(K)$, μ Haar measure on Γ , G the Zariski closure of Γ in GL_n , and f an element of the coordinate ring of GL_n over K which does not vanish identically on any component of G . Then for all $\epsilon > 0$, there exists α in the value group of V such that*

$$\mu(\{\gamma \in \Gamma \mid v(f(\gamma)) > \alpha\}) < \epsilon.$$

Proof. We say that a K -subvariety $X \subset G$ is *thin* if for every finite subset $\{f_1, \dots, f_m\}$ of the coordinate ring A of GL_n such that $V(f_1, \dots, f_m) \cap G = X$, we have

$$\lim_{\alpha \rightarrow \infty} \mu(\{\gamma \in \Gamma \mid \forall i \ v(f_i(\gamma)) > \alpha\}) = 0.$$

The proposition follows immediately from the more general statement that every subvariety X of G of codimension ≥ 1 is thin. We prove this by Noetherian induction; the induction step consists in proving that X is thin if all of its proper subvarieties are so.

The base case is the empty variety: X is empty if and only if in the coordinate ring $B = A/I$ of G , $(\bar{f}_1, \dots, \bar{f}_m) = B$ or, in other words, if and only if there exist $a_1, \dots, a_m \in A$ such that

$$\bar{a}_1 \bar{f}_1 + \dots + \bar{a}_m \bar{f}_m = 1.$$

In particular,

$$a_1(\gamma)f_1(\gamma) + \dots + a_m(\gamma)f_m(\gamma) = 1$$

for all $\gamma \in \Gamma$. Applying v , we obtain

$$\min_{1 \leq i \leq m} v(a_i(\gamma)) + v(f_i(\gamma)) \leq 0.$$

On the other hand, the sets

$$\Gamma_{i,\alpha} := \{\gamma \in \Gamma \mid v(a_i(\gamma)) \geq -\alpha\}$$

are open, so by compactness, there exists α in the value group of V with $v(a_i(\gamma)) \geq -\alpha$ for all i , and for all $\gamma \in \Gamma$. Thus,

$$\{\gamma \in \Gamma \mid \forall i \ v(f_i(\gamma)) > \alpha\} = \emptyset.$$

For the induction step, note that Γ acts on GL_n by left-translation and therefore acts on A . We write f^γ for the image of $f \in A$ by $\gamma \in \Gamma$. For $x \in X$ and $\gamma_1, \dots, \gamma_i \in \Gamma$, the set $\{g \in G \mid gx \in \gamma_1 X \cup \dots \cup \gamma_i X\}$ has dimension $\dim X < \dim G$. As Γ is dense in G , there exists an infinite sequence $\gamma_1, \gamma_2, \dots \in \Gamma$ such that $\gamma_i X \neq \gamma_j X$ for all $i \neq j$. By the induction hypothesis, $\gamma_j^{-1} \gamma_i X \cap X$ is thin for all $i \neq j$. Let (f_1, \dots, f_k) be an ideal in A such that $V(f_1, \dots, f_k) \cap G = X$. Then

$$V(f_1, \dots, f_k, f_1^{\gamma_j^{-1} \gamma_i}, \dots, f_k^{\gamma_j^{-1} \gamma_i}) \cap G = \gamma_j^{-1} \gamma_i X \cap X,$$

so for all positive integers n there exists α such that

$$\mu(\{\gamma \in \Gamma \mid \forall h \leq k, \ v(f_h(\gamma)) > \alpha, v(f_h^{\gamma_j^{-1} \gamma_i}(\gamma)) > \alpha\}) < n^{-2}$$

for all $1 \leq i < j \leq n$. Translating by γ_j ,

$$\mu(\{\gamma \in \Gamma \mid \forall h \leq k, \ v(f_h^{\gamma_j}(\gamma)) > \alpha, v(f_h^{\gamma_i}(\gamma)) > \alpha\}) < n^{-2}. \quad (1)$$

Let

$$S_j = \{\gamma \in \Gamma \mid \forall h \leq k, \ v(f_h^{\gamma_j}(\gamma)) > \alpha\}.$$

By inclusion-exclusion,

$$1 - \sum_{1 \leq i \leq n} \mu(S_i) + \sum_{1 \leq i < j \leq n} \mu(S_i \cap S_j) \geq 0.$$

As

$$\mu(S_j) = \mu(\{\gamma \in \Gamma \mid \forall h \leq k, \ v(f_h(\gamma)) > \alpha\})$$

for all j , (1) implies

$$\mu(\{\gamma \in \Gamma \mid \forall h \leq k, \ v(f_h(\gamma)) > \alpha\}) \leq \frac{1 + \binom{n}{2} n^{-2}}{n} < \frac{2}{n},$$

where n can be taken to be as large as we wish. The proposition follows by induction. \square

As an immediate consequence of Proposition 2.3 above and the classical Cebotarev density theorem we have:

Theorem 2.4 *Let F be a number field, \overline{F} an algebraic closure, and $G_F = \text{Gal}(\overline{F}/F)$. Let V be a valuation ring with fraction field K and $\rho: G_F \rightarrow \text{GL}_n(K)$ be a continuous, finitely ramified representation. Let X be a conjugation-invariant subvariety of positive codimension in each component of the Zariski closure of $\rho(G_F)$. Then the set of primes of F at which the Frobenii under ρ lie in X has Dirichlet density 0.*

In the case when ρ is finitely ramified, we do not know if quantitative refinements of the density 0 result, analogous to Théorème 10 of [S2], are true in this general setting. In any case our “soft techniques” will not yield such quantitative refinements. The theorem of the next section allows us to get rid of the finitely ramified hypothesis in the theorem: but without that assumption quantitative refinements cannot be expected as the last section of [KLR] shows.

2.2 Ramification

Here is the main theorem of this section.

Theorem 2.5 *Let F be a number field, \overline{F} an algebraic closure, and $G_F = \text{Gal}(\overline{F}/F)$. Let V be a valuation ring with fraction field K where K is complete of characteristic zero and residue characteristic ℓ . Let $\rho: G_F \rightarrow \text{GL}_n(K)$ be a continuous semisimple representation. Then the set of primes of F at which ρ is ramified has Dirichlet density 0.*

To prove the main theorem, we need two simple lemmas.

Lemma 2.6 *Let $q > 1$ be a positive integer, L an algebraically closed field, and x and y elements of $\text{GL}_n(L)$ such that $xyx^{-1} = y^q$. Then either y is semisimple and of finite order in $\text{GL}_n(L)$ or there exist eigenvalues λ_1 and λ_2 of x with $\lambda_1/\lambda_2 = q$.*

Proof. Let $y = y_s y_u$ be the multiplicative Jordan decomposition. Then

$$xy_s x^{-1} = y_s^q, \quad xy_u x^{-1} = y_u^q.$$

If the characteristic polynomial of y_s is $\prod_{i=1}^n x - \lambda_i$, then there exists a permutation $\pi \in S_n$ such that

$$\lambda_i^q = \lambda_{\pi(i)},$$

which means that

$$\lambda_i^{\prod_{j=1}^n q^j - 1} = 1.$$

Thus either y is semisimple of finite order or $y_u \neq 1$.

In the latter case, x and y_u generate a solvable subgroup and therefore a subgroup of $B(L)$, where B is the Borel subgroup stabilizing some maximal flag of L^n . Let

$$U_1 = [B, B], U_2 = [B, U_1], \dots, U_n = \{1\}$$

denote the descending central series. Thus $y_u \in U_1$ but $y_u \notin U_n$. Choose k so that $y_u \in U_k \setminus U_{k+1}$. Then $B(L)/U_1(L)$ acts on U_k/U_{k+1} , and \bar{y}_u is an eigenvector of $\bar{x} \in B(L)/U_1(L)$ with eigenvalue q ; as the eigenvalues of the diagonal matrix $\sum \lambda_i e_{ii}$ acting on $U_k(L)/U_{k+1}(L)$ are

$$\lambda_i \lambda_{i+k}^{-1}, \quad 1 \leq i \leq n - k,$$

the lemma follows. \square

Lemma 2.7 *In every valuation ring V , 1 has an open neighborhood in which the only root of unity is 1 itself.*

Proof. For $\alpha > 0$ in the value group of V , $v(x-1) \geq \alpha$ implies $v(x^n-1) \geq \alpha$. It therefore suffices to find α such that for every prime p , and every primitive p th root of unity ζ_p , $v(\zeta_p - 1) \leq \alpha$. As $\zeta_p - 1$ divides p , $v(\zeta_p - 1) \leq v(p)$. There is at most one rational prime p in V for which $v(p) > 0$, so we can take

$$\alpha := \sup_p v(p) < \infty.$$

\square

Proof of Theorem 2.5. The representation ρ can be wildly ramified at \wp only if \wp has the same residue characteristic as V ; henceforth we will ignore this finite set of primes. If \wp is a prime of F and G_\wp and T_\wp denote the Galois group of the maximal tamely ramified extension of the completion F_\wp and the tame inertia subgroup respectively, we have a short exact sequence

$$0 \rightarrow T_\wp \rightarrow G_\wp \rightarrow \hat{\mathbb{Z}} \rightarrow 0.$$

The quotient group $\hat{\mathbb{Z}}$ is topologically generated by the Frobenius class σ_\wp , T_\wp is topologically generated by some non-canonical class τ_\wp , and

$$\sigma_\wp \tau_\wp \sigma_\wp^{-1} = \tau_\wp^{\|\wp\|}. \quad (2)$$

By Lemma 2.6 and Lemma 2.7, there exists an open neighborhood $G_{F'}$ of the identity in G_F such that no element of $\rho(G_{F'})$ can have an eigenvalue which is a non-trivial root of unity. We exclude henceforth from discussion the finite set of primes \wp' in the finite extension F' which are ramified over F . Thus the natural maps $G_{\wp'} \rightarrow G_\wp$ restrict to isomorphisms $T_{\wp'} \rightarrow T_\wp$. It follows that $\rho(\tau_\wp)$ is unipotent.

Let G be the Zariski-closure of $\rho(G_F)$ and Z the center of the identity component G° . Choose a faithful K -representation $G/Z \hookrightarrow \mathrm{GL}_m$. As 1 is the only unipotent element in $Z(K)$, T_\wp lies in the kernel of the composition map $G_F \rightarrow G(K) \rightarrow (G/Z)(K) \rightarrow \mathrm{GL}_m(K)$ if and only if ρ is unramified at \wp . Replacing ρ by this composition if necessary, we assume without loss of generality that G has semisimple identity component.

By (2) and Lemma 2.6, either ρ is unramified at \wp or $\rho(\sigma_\wp)$ has two eigenvalues in \bar{K} whose ratio is $\|\wp\|$. Let ε denote the ℓ -adic cyclotomic character. Consider the direct sum

$$\alpha = \rho \oplus \varepsilon: G_F \rightarrow G \times \mathrm{GL}_1,$$

and let H denote the Zariski closure of $\alpha(G_F)$ in this representation. Thus $H \subset G \times \mathrm{GL}_1$ and H projects onto each factor. By Goursat's lemma, H is the pullback of the graph of an isomorphism between a quotient of G and a quotient of GL_1 . Every quotient of GL_1 is a torus and G admits no non-trivial toric quotient, so $H = G \times \mathrm{GL}_1$. Let $X \subset H$ denote the subvariety of pairs $(g, c) \in H$ such that g and gc have at least one eigenvalue in common. For each g there are only finitely many possible values of c , so X is of codimension ≥ 1 in each component of H . By the construction of X , if ρ is ramified at

\wp , then for any choice of σ_\wp , $\alpha(\sigma_\wp)$ lies in X . By Proposition 2.3 (with μ Haar measure on $\rho(G_F)$), for any $\epsilon > 0$, we can find an open and closed neighborhood N_ϵ of $X(K)$ in $H(K)$ such that $\mu(\rho(G_F) \cap N_\epsilon) < \epsilon$. Let F'' be a finite Galois extension of F such that $\alpha^{-1}(N_\epsilon)$ is a finite union of $G_{F''}$ cosets. Thus the image of $\alpha^{-1}(X(K))$ in $\text{Gal}(F''/F)$ has less than $\epsilon |\text{Gal}(F''/F)|$ elements and is a union of conjugacy classes in $\text{Gal}(F''/F)$ (since X is a union of conjugacy classes in H). By the Chebotarev density theorem, there is a set of primes \wp of F of Dirichlet density at least $1 - \epsilon$ such that ρ is unramified at \wp . The theorem follows. \square

Remarks: 1) As almost all ramification is unipotent, the proof also shows that if ρ has abelian image it is *finitely ramified*.

2) Just as in [Kh] we can define the notion of a converging sequence of \mathbb{C}_ℓ valued Galois representations $\rho_i : G_F \rightarrow \text{GL}_n(\mathbb{C}_\ell)$ in the residually irreducible case, i.e., $\text{tr}(\rho_i(g))$ is a Cauchy sequence uniformly for $g \in G_F$. (It is an immediate consequence of [Ca] that for each n , the ρ_i 's mod ℓ^n are then eventually constant, and thus have a limit ρ .) Just as in [Kh] one proves by essentially the same method as above that the Dirichlet density of primes that ramify in any of the representations ρ_i is 0.

3 An example

For a finite field \mathbb{F}_{ℓ^d} , we denote the ring of Witt vectors of \mathbb{F}_{ℓ^d} by $W(\mathbb{F}_{\ell^d})$. We assume $\ell > 3$ in this section.

Lemma 3.1 *Let $\ell > 3$ and $B_m \subset \text{GL}_2(W(\mathbb{F}_{\ell^d})/l^m W(\mathbb{F}_{\ell^d}))$ be a subgroup whose mod ℓ reduction is $\text{GL}_2(\mathbb{F}_\ell)$. Then B_m contains (up to conjugation by an element of the form $I + \ell X \in \text{GL}_2(W(\mathbb{F}_{\ell^d})/l^m W(\mathbb{F}_{\ell^d}))$) an element $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ where $a \in (\mathbb{Z}/l^m \mathbb{Z})^*$ and $a \not\equiv \pm 1 \pmod{\ell}$.*

Proof. By hypothesis we know that the mod ℓ reduction of B_m contains an element h of the form $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ where $a \in (\mathbb{Z}/l^m \mathbb{Z})^*$ and $a \not\equiv \pm 1 \pmod{\ell}$. Choose any lift $g \in B_m$ of this mod ℓ element h . As the characteristic polynomial of h has distinct roots in \mathbb{F}_ℓ , and as $\ell > 2$, the element g can be conjugated to a diagonal matrix viewed as an element of $\text{GL}_2(W(\mathbb{F}_{\ell^d})/l^m W(\mathbb{F}_{\ell^d}))$.

By taking ℓ^r th powers, for large enough r , we obtain an element that has a conjugate in B_m with the desired properties. \square

Remark: We need Lemma 3.1 as an ingredient for Fact 3.6. In [KLR] our ρ_m always had image $\mathrm{GL}_2(\mathbb{Z}/\ell^m\mathbb{Z})$ and it obviously contained elements like those in Lemma 3.1.

Definition 3.2 *We say a Galois representation $\bar{\rho}$ satisfies our running hypotheses if*

- $\bar{\rho} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z})$ is surjective with $\ell \geq 5$,
- $\det(\bar{\rho}) = \varepsilon$, the cyclotomic character, and
- $\bar{\rho}$ is unramified outside a finite set S of primes that necessarily contains ℓ .

In this section we assume, primarily for simplicity, that *all determinants are the cyclotomic character ε* . We denote by $Ad^0\bar{\rho}$ the Galois module of 2×2 trace zero matrices over \mathbb{F}_ℓ with Galois action through $\bar{\rho}$ via conjugation. We denote by $(Ad^0\bar{\rho})^*$ the Cartier dual of $Ad^0\bar{\rho}$. We recall some Definitions, Facts, Lemmas and Propositions from [KLR]. The precise [KLR] references are given in each case parenthetically.

Definition 3.3 *(A slight variant of Definition 1 of [KLR]) Suppose $\bar{\rho}$ satisfies our running hypotheses. We say a prime q is nice (for $\bar{\rho}$) if*

- q is not $\pm 1 \pmod{\ell}$,
- $\bar{\rho}$ is unramified at q ,
- the eigenvalues of $\bar{\rho}(\sigma_q)$ (where σ_q is Frobenius at q) have ratio q .

Let ρ_m be a deformation of $\bar{\rho}$ to $\mathrm{GL}_2(W(\mathbb{F}_{\ell^d})/\ell^m W(\mathbb{F}_{\ell^d}))$ with determinant the cyclotomic character. We say a prime q is ρ_m -nice if

- q is nice for $\bar{\rho}$,
- ρ_m is unramified at q , and the (necessarily distinct) roots of the characteristic polynomial of $\rho_m(\sigma_q)$ have ratio q . Note that since q is nice, the mod ℓ^m characteristic polynomial of $\rho_m(\sigma_q)$ has distinct roots that are units; it follows that the eigenvalues of $\rho_m(\sigma_q)$ are well-defined in $W(\mathbb{F}_{\ell^d})/\ell^m W(\mathbb{F}_{\ell^d})$.

Fact 3.4 (Weak version of Fact 5 of [KLR]) Suppose $\bar{\rho}$ satisfies our running hypotheses. Let the sets $\{f_1, \dots, f_n\}$ and $\{\phi_1, \dots, \phi_r\}$ be linearly independent in $H^1(\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \text{Ad}^0 \bar{\rho})$ and $H^1(\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), (\text{Ad}^0 \bar{\rho})^*)$ respectively. Let $\mathbb{Q}(\text{Ad}^0(\bar{\rho}))$ be the field fixed by the kernel of the action of $G_{\mathbb{Q}}$ on $\text{Ad}^0(\bar{\rho})$. Let $\mathbf{K} = \mathbb{Q}(\text{Ad}^0(\bar{\rho}), \mu_\ell)$ be the field obtained by adjoining the ℓ th roots of unity to $\mathbb{Q}(\text{Ad}^0(\bar{\rho}))$. We denote by \mathbf{K}_{f_i} and \mathbf{K}_{ϕ_j} the fixed fields of the kernels of the restrictions of f_i and ϕ_j to $G_{\mathbf{K}}$, the absolute Galois group of \mathbf{K} . Then, as $\mathbb{Z}/\ell\mathbb{Z}[\text{Gal}(\mathbf{K}/\mathbb{Q})]$ -modules, $\text{Gal}(\mathbf{K}_{f_i}/\mathbf{K})$ and $\text{Gal}(\mathbf{K}_{\phi_j}/\mathbf{K})$ are isomorphic, respectively, to $\text{Ad}^0 \bar{\rho}$ and $(\text{Ad}^0 \bar{\rho})^*$ and each of the fields \mathbf{K}_{f_i} and \mathbf{K}_{ϕ_j} is linearly disjoint over \mathbf{K} with the compositum of the others. Let I be a subset of $\{1, \dots, n\}$ and J a subset of $\{1, \dots, r\}$. Then there exists a Chebotarev set X (a set X of primes of positive density coming from an application of Chebotarev's Theorem) of primes $w \notin S$ such that

- w is nice,
- $f_i|_{G_w} \neq 0$ for $i \in I$ and $f_i|_{G_w} = 0$ for $i \in \{1, \dots, n\} \setminus I$,
- $\phi_j|_{G_w} \neq 0$ for $j \in J$ and $\phi_j|_{G_w} = 0$ for $j \in \{1, \dots, r\} \setminus J$.

Fact 3.5 (Lemma 6 of [KLR]) Denote by the symbol $\text{III}_X^i(M)$ the kernel of the localisation map $H^1(G_X, M) \rightarrow \bigoplus_{v \in X} H^i(G_v, M)$. Let $\bar{\rho}$ satisfy our running hypotheses. There exists a finite set T of nice primes such that $\text{III}_{S \cup T}^1(\text{Ad}^0 \bar{\rho})$ and $\text{III}_{S \cup T}^2(\text{Ad}^0 \bar{\rho})$ are trivial. After enlarging S , we may thus assume $\text{III}_S^1(\text{Ad}^0 \bar{\rho})$ and $\text{III}_S^2(\text{Ad}^0 \bar{\rho})$ are trivial.

Remark: While Fact 3.5 above requires Fact 3.4, a study of the proofs of Lemma 3.7 and Propositions 3.8 and 3.9 (in [KLR]) shows they require only Fact 3.6 below.

Fact 3.6 (Weak variant of Fact 5 of [KLR]) Let $\bar{\rho}$ satisfy our running hypotheses. Let ρ_m be a deformation of $\bar{\rho}$ to $\text{GL}_2(W(\mathbb{F}_{\ell^d})/\ell^m W(\mathbb{F}_{\ell^d}))$ unramified outside S with $\det(\rho_m) = \varepsilon$. Suppose $\{\phi_1, \dots, \phi_r\}$ is independent in $H^1(\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), (\text{Ad}^0 \bar{\rho})^*)$. Let J be a subset of $\{1, \dots, r\}$. Then there exists a Chebotarev set X (a set X of primes of positive density coming from an application of Chebotarev's Theorem) of primes $w \notin S$ such that

- w is ρ_m -nice,
- $\phi_j|_{G_w} \neq 0$ for $j \in J$ and $\phi_j|_{G_w} = 0$ for $j \in \{1, \dots, r\} \setminus J$.

Proof. By Lemma 3.1 we know that the image of ρ_m contains (up to conjugation) an element an element $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ with $a \in (\mathbb{Z}/\ell^m\mathbb{Z})^*$ and $a \not\equiv \pm 1 \pmod{\ell}$. Using the fact that $\det(\rho_m) = \varepsilon$, one sees from the proof of Fact 5 of [KLR] that the Chebotarev set of primes with Frobenius in the conjugacy class of this element and the desired splitting properties in the fields \mathbf{K}_{ϕ_i} provides the ρ_m -nice primes. \square

Remark: It is not true that one can also include in Fact 3.6 splitting properties with independent elements of $H^1(\text{Gal}(\mathbb{Q}/\mathbb{Q}), \text{Ad}^0 \bar{\rho})$. Such a statement is true for $\text{GL}_2(W(\mathbb{F}_{\ell^d})/\ell^m W(\mathbb{F}_{\ell^d}))$ representations provided the minimal field of definition of the adjoint representation is \mathbb{F}_{ℓ^d} . In our situation the minimal field of definition of the adjoint representation is always \mathbb{F}_{ℓ} .

Lemma 3.7 (Lemma 8 of [KLR]) *Let $\bar{\rho}$ satisfy our running hypotheses. Let ρ_m be a deformation of $\bar{\rho}$ to $\text{GL}_2(W(\mathbb{F}_{\ell^d})/\ell^m W(\mathbb{F}_{\ell^d}))$ unramified outside a set S . Suppose $\text{III}_S^1(\text{Ad}^0 \bar{\rho})$ and $\text{III}_S^2(\text{Ad}^0 \bar{\rho})$ are trivial and $\det(\rho_m) = \varepsilon$. Let R be any finite collection of unramified primes of ρ_m disjoint from S . Then there is a finite set $Q = \{q_1, \dots, q_n\}$ of ρ_m -nice primes disjoint from $R \cup S$ such that the maps*

$$H^1(G_{S \cup R \cup Q}, (\text{Ad}^0 \bar{\rho})^*) \rightarrow \bigoplus_{v \in Q} H^1(G_v, (\text{Ad}^0 \bar{\rho})^*), \quad (3)$$

$$H^1(G_{S \cup R \cup Q}, \text{Ad}^0 \bar{\rho}) \rightarrow \bigoplus_{v \in S \cup R} H^1(G_v, \text{Ad}^0 \bar{\rho}) \quad (4)$$

and

$$H^1(G_{S \cup Q}, \text{Ad}^0 \bar{\rho}) \rightarrow \left(\bigoplus_{v \in S} H^1(G_v, \text{Ad}^0 \bar{\rho}) \right) \oplus \left(\bigoplus_{r \in R} H_{nr}^1(G_r, \text{Ad}^0 \bar{\rho}) \right) \quad (5)$$

are isomorphisms of \mathbb{F}_{ℓ} vector spaces. ($H_{nr}^1(G_v, M)$ denotes the image of the inflation map $H^1(G_v/I_v, M^{I_v}) \rightarrow H^1(G_v, M)$). Upon tensoring with \mathbb{F}_{ℓ^d} they can be viewed as isomorphisms of \mathbb{F}_{ℓ^d} vector spaces.

Proof. That the deformation to the Witt ring is the only difference between here and the [KLR] situation. Fact 3.6 provides us with the necessary ρ_m -nice primes for the proof of the Proposition. The last statement about tensoring with \mathbb{F}_{ℓ^d} is obvious. \square

Proposition 3.8 (*Proposition 9 of [KLR]*) *Following the notation of Lemma 3.7, let A be any finite set of primes disjoint from $S \cup R \cup Q$. Fix k between 1 and n . There exists a Cebotarev set T_k of primes t_k such that*

- *all $t_k \in T_k$ are ρ_m nice,*
- *for any $t_k \in T_k$, the kernel of the map of \mathbb{F}_ℓ vector spaces*

$$H^1(G_{S \cup Q \cup \{t_k\}}, Ad^0 \bar{\rho}) \rightarrow \oplus_{v \in S} H^1(G_v, Ad^0 \bar{\rho}) \oplus_{v \in R} H_{nr}^1(G_v, Ad^0 \bar{\rho}) \quad (6)$$

is one dimensional spanned by f_{t_k} ,

- *$f_{t_k}|_{G_v} = 0$ for all $v \in S \cup R \cup Q \cup A \setminus \{q_k\}$,*
- *f_{t_k} is unramified at G_{q_k} and $f_{t_k}(\sigma_{q_k}) \neq 0$.*

Proof. The only difference between this statement and that of [KLR] is that the deformation is to the Witt vectors. The proof carries over word for word. \square

Proposition 3.9 (*Proposition 10 of [KLR]*) *Let the notations be as in Lemma 3.7 and Proposition 3.8. There is a set \tilde{T}_k of one or two primes of T_k such that*

- *there is a linear combination f_k of the elements f_{t_k} for $t_k \in \tilde{T}_k$ such that $f_k(\sigma_{t_k}) = 0$ for all $t_k \in \tilde{T}_k$ and $f_k|_{G_{q_k}} \neq 0$,*
- *$j < k$ implies that for $t_j \in \tilde{T}_j$ we have $f_{t_k}(\sigma_{t_j}) = 0$,*
- *$f_k|_{G_v} = 0$ for all $v \in S \cup R \cup Q \setminus \{q_k\}$.*

Proof. As in Proposition 3.8, the proof from [KLR] carries over word for word. \square

Remarks: 1) If ρ_m is a deformation of $\bar{\rho}$ to $\mathrm{GL}_2(W(\mathbb{F}_{\ell^d})/l^m W(\mathbb{F}_{\ell^d}))$ and $f \in H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mathrm{Ad}^0 \bar{\rho} \otimes_{\mathbb{F}_{\ell}} \mathbb{F}_{\ell^d})$ then $(I + \ell^{m-1} f) \rho_m$ is also a deformation of $\bar{\rho}$ to $\mathrm{GL}_2(W(\mathbb{F}_{\ell^d})/l^m W(\mathbb{F}_{\ell^d}))$ that is congruent to $\rho_m \bmod \ell^{m-1}$.
2) In [KLR] f_k of Proposition 3.9 (Proposition 10 of [KLR]) spanned a well defined line (over \mathbb{F}_{ℓ}). Given ρ_m to $\mathrm{GL}_2(\mathbb{Z}/\ell^m \mathbb{Z})$, we proved in [KLR] that there was a suitable \mathbb{F}_{ℓ} linear combination h of the of the f_k and an element of $H^1(G_{S \cup Q \cup R}, \mathrm{Ad}^0 \bar{\rho})$ such that $(I + \ell^{m-1} h) \rho_m$ had previously chosen characteristic polynomials at primes of R , and was unobstructed at the primes of $S \cup Q \cup (\cup_k T_k)$. Here we will use an $\mathbb{F}_{\ell^{2^{m-1}}}$ linear combination.

Theorem 3.10 *There exists a potentially semistable 2-dimensional continuous irreducible Galois representation $\rho: G_{\mathbb{Q}} := \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathrm{GL}_2(\mathbb{C}_{\ell})$ ($\ell \geq 5$) such that the trace of Frobenius $\mathrm{tr}(\sigma_r)$ belongs to $\bar{\mathbb{Q}}_{\ell} \subset \mathbb{C}_{\ell}$ for all r in a set of primes of \mathbb{Q} of density 1 but ρ is not conjugate to any representation with values in $\bar{\mathbb{Q}}_{\ell}$.*

Proof. The proof follows the general strategy of the proof of Application I of [KLR], but in addition needs the refinements above.

- Let $\bar{\rho}$ satisfy our running hypotheses. We call this representation ρ_1 from now on.
- Use Fact 3.5 to enlarge S to S_1 for which $\mathrm{III}_{S_1}^1(\mathrm{Ad}^0 \bar{\rho})$ and $\mathrm{III}_{S_1}^2(\mathrm{Ad}^0 \bar{\rho})$ are trivial and obstructions to lifting can be detected locally.
- Once and for all choose for each $v \in S_1$ local deformations of $\rho_1|_{G_v}$ to $\mathrm{GL}_2(\mathbb{Z}_{\ell})$. For $v = \ell$ we choose a potentially semistable deformation. See [R3] for a proof that these local deformations exist.
- Since all local mod ℓ representations admit deformations to mod ℓ^2 (see above) and all global obstructions to deforming to mod ℓ^2 can be detected locally, we can deform ρ_1 to $\rho_2: \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathrm{GL}_2(\mathbb{Z}/\ell^2 \mathbb{Z})$ with $\det(\rho_2) = \varepsilon$. We are in a situation where we can apply Lemma 3.7 and Propositions 3.8 and 3.9.
- Choose a finite set R_2 of unramified primes in ρ_2 beneath some bound, say b_2 . The actual value of b_2 (and our later bounds b_3, b_4, \dots) is not important. What matters is that R_2 contains at least one element that we call r_2 .

Since all determinants of all deformations will be the cyclotomic character, choosing the characteristic polynomial of an unramified prime amounts to choosing its trace. Thus henceforth we will only speak about choosing the traces of Frobenii.

Our ρ_2 is *not* the mod ℓ^2 representation we need, so we will alter it by a certain 1-cohomology class. Once and for all we choose traces in $W(\mathbb{F}_\ell)$ for all primes in $R_2 \setminus \{r_2\}$. For r_2 we choose the trace in $W(\mathbb{F}_{\ell^2}) \setminus W(\mathbb{F}_\ell)$. For all $r \in R_2$ these traces are necessarily congruent to $\text{Trace}(\rho_2(\sigma_r)) \pmod{\ell}$.

- We use Lemma 3.7 to adjust ρ_2 by a 1-cohomology class g_2 with coefficients necessarily in $Ad^0 \bar{\rho}_{/\mathbb{F}_\ell} \otimes_{\mathbb{F}_\ell} \mathbb{F}_{\ell^2}$ so that the traces of Frobenii of all primes in R_2 are $(\pmod{\ell^2})$ as previously picked *and* so that at the places of S_1 the local representations are the mod ℓ^2 reduction of the previously chosen characteristic zero local representations. Thus there are no obstructions to deforming to mod ℓ^3 at primes of S_1 . Our new representation is $(I + \ell g_2)\rho_2$.

The representation $(I + \ell g_2)\rho_2$ may be ramified at new primes in the finite set Q_2 of Lemma 3.7. There may be obstructions at primes of Q_2 to deforming to mod ℓ^3 .

- We use Proposition 3.9 to remove obstructions at primes of Q_2 , adjusting by a 1-cohomology class with coefficients in $Ad^0 \bar{\rho}_{/\mathbb{F}_\ell} \otimes_{\mathbb{F}_\ell} \mathbb{F}_{\ell^2}$ while introducing new ramified primes in a finite set T_2 at which *there are no obstructions* to deforming to mod ℓ^3 . The trace of Frobenius of all primes in R_2 will be as previously chosen. The new map, which we yet again call ρ_2 , has image in $\text{GL}_2(W(\mathbb{F}_{\ell^2})/\ell^2 W(\mathbb{F}_{\ell^2}))$. Let $S_2 = S_1 \cup Q_2 \cup T_2$ be the set of ramified primes of this new deformation. For each $v \in S_2 \setminus S_1$ once and for all choose local deformations of (our new) $\rho_2|_{G_v}$ to $\text{GL}_2(W(\mathbb{F}_{\ell^2}))$. (We have no control over whether $\rho_2|_{G_v}$ is ramified. If $\rho_2|_{G_v}$ is unramified, we may choose our characteristic zero deformation to be either unramified or ramified as we please.)

Proposition 3.9 (and its proof) guarantees we can do this. (See [KLR] for details).

- Deform ρ_2 to $\rho_3 : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(W(\mathbb{F}_{\ell^2})/\ell^3 W(\mathbb{F}_{\ell^2}))$.

- Now fix a set $R_3 \supset R_2$ of unramified primes in ρ_3 beneath some bound b_3 . The traces of Frobenii of these primes are determined mod ℓ^2 by ρ_2 . Fix a prime r_3 in $R_3 \setminus R_2$.

Our ρ_3 is *not* the mod ℓ^3 representation we need, so we will alter it by a certain 1-cohomology class. Once and for all we choose traces in $W(\mathbb{F}_{\ell^2})$ for all primes r in $R_3 \setminus (R_2 \cup \{r_3\})$. For r_3 we choose the trace in $W(\mathbb{F}_{\ell^4}) \setminus W(\mathbb{F}_{\ell^2})$. For $r \in R_3 \setminus R_2$ all these traces are necessarily congruent mod ℓ^2 to $\text{Trace}(\rho_2(\sigma_r))$.

- Use Lemma 3.7 to adjust ρ_3 by an 1-cohomology class g_3 with coefficients necessarily in $\text{Ad}^0 \bar{\rho}_{/\mathbb{F}_\ell} \otimes_{\mathbb{F}_\ell} \mathbb{F}_{\ell^4}$ so that the traces of Frobenii of all primes in R_3 are (mod ℓ^3) as previously picked *and* so that at the places of S_2 the local representations are the mod ℓ^3 reduction of the previously chosen characteristic zero local representations. Thus there are no obstructions to deforming to mod ℓ^4 at primes of S_2 . This new representation is $(I + \ell^2 g_3)\rho_3$.

The representation $(I + \ell^2 g_3)\rho_3$ may be ramified at new primes in a finite set Q_3 . There may be obstructions at primes of Q_3 to deforming to mod ℓ^4 .

- Use Proposition 3.9 to remove obstructions at primes of Q_3 while introducing new ramified primes in a finite set T_3 at which *there are no obstructions* to deforming to mod ℓ^4 . The trace of Frobenius of all primes in R_3 will be as previously chosen. Let $S_3 = S_2 \cup Q_3 \cup T_3$ be the set of ramified primes of this new ρ_3 . For each $v \in S_3 \setminus S_2$ once and for all choose local deformations of (our new) $\rho_3|_{G_v}$ to $\text{GL}_2(W(\mathbb{F}_{\ell^4}))$. (We have no control over whether (our new) $\rho_3|_{G_v}$ is ramified. If (our new) $\rho_3|_{G_v}$ is unramified, we may choose our characteristic zero deformation to be either unramified or ramified as we please.) Proposition 3.9 (and its proof) guarantees we can do this. (See [KLR] for details).
- At the m th stage we will have a representation

$$\rho_m : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(W(\mathbb{F}_{\ell^{2^{m-1}}})/\ell^m W(\mathbb{F}_{\ell^{2^{m-1}}}))$$

that we will be able to deform to

$$\rho_{m+1} : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(W(\mathbb{F}_{\ell^{2^{m-1}}})/\ell^{m+1} W(\mathbb{F}_{\ell^{2^{m-1}}})) .$$

We make sure $R_{m+1} \setminus R_m$ has at least one prime r_{m+1} whose trace of Frobenius we choose in $W(\mathbb{F}_{\ell^{2^m}}) \setminus W(\mathbb{F}_{\ell^{2^{m-1}}})$ and then using Lemma 3.7 alter ρ_{m+1} in by a suitable 1-cohomology class g_{m+1} with coefficients in $Ad^0 \bar{\rho} \otimes_{\mathbb{F}} \mathbb{F}_{\ell^{2^m}}$. Then $(I + \ell^m g_{m+1})\rho_{m+1}$ will have the trace of Frobenius of all primes in R_{m+1} as previously chosen *and* at all places of S_m the local representations are the mod ℓ^{m+1} reduction of the previously chosen characteristic zero local representations. Thus there are no obstructions to deforming to mod ℓ^{m+2} at primes of S_m . This process introduces ramification at new primes and we denote the set of these and previously ramified primes by S_{m+1} . Proposition 3.9 allows us, once and for all, to choose appropriate characteristic zero deformations of $\rho_{m+1}|_{G_v}$ for $v \in S_{m+1} \setminus S_m$. We may continue the inductive deformation process.

The inverse limit of the compatible system of mod ℓ^m representations is valued in $\mathrm{GL}_2(W(\bar{\mathbb{F}}_\ell))$. By the choices of the traces of the Frobenii of r_i this representation is *not* valued in $\mathrm{GL}_2(W(\mathbb{F}_{\ell^d}))$ for any d . Theorem 2.5 and the Baire category theorem show this is an example of a representation that has the properties asserted in the theorem. \square

4 Fields of definition of finitely ramified representations

We show that in examples like the one above the representation is necessarily infinitely ramified (although in the previous example we could have simply ensured that during the construction).

Theorem 4.1 *Let G be a topologically finitely generated profinite group, and $\rho : G \rightarrow \mathrm{GL}_n(\mathbb{C}_\ell)$ a continuous homomorphism. Let $\{x_i\} \subset G$ be a set of elements such that the union of their conjugacy classes is dense in G and such that $\mathrm{tr}(\rho(x_i)) \in \bar{\mathbb{Q}}_\ell$ for all $x_i \in X$. Then there exists a finite extension L of \mathbb{Q}_ℓ such that $\mathrm{tr}(\rho(g)) \in L$ for all $g \in G$. If ρ is semisimple, replacing L by a finite extension if necessary, $\rho(G)$ can be conjugated into a subgroup of $\mathrm{GL}_n(L)$.*

Proof. By Lemma 2.2 above ρ has a model over \mathcal{O} the valuation ring of \mathbb{C}_ℓ , and thus we consider ρ as taking values in $\mathrm{GL}_n(\mathcal{O})$. By continuity of ρ ,

for any m , $\rho \bmod \ell^m$ has finite image, and can be regarded as taking values in $\mathrm{GL}_n(\mathcal{O}_{L_m}/\ell^m \mathcal{O}_{L_m})$ where \mathcal{O}_{L_m} is the ring of integers of a finite extension L_m of \mathbb{Q}_ℓ . This can be all be arranged because of the surjectivity of the map $\bar{\mathbb{Z}}_\ell \rightarrow \mathcal{O}/\ell^m \mathcal{O}$. The residual homomorphism that arises from reducing ρ modulo the maximal ideal of \mathcal{O} is a homomorphism $\bar{\rho} : G_F \rightarrow \mathrm{GL}_n(k)$ for a k/\mathbb{F}_ℓ a finite extension of \mathbb{F}_ℓ , and we take k to be the minimal such field (i.e., it is generated by the finitely many matrix entries of the images of $\rho \bmod$ the maximal ideal of \mathcal{O}). We assume for convenience that L_i are Galois over \mathbb{Q}_ℓ , and also that $L_m \subset L_{m+1}$. We also assume that L_1 is an unramified extension of \mathbb{Q}_ℓ with residue field k .

Consider the category \mathcal{C} of complete local Noetherian $W(k)$ -algebras with residue field k , and with morphisms in this category, local morphisms of $W(k)$ -algebras, such that the induced map on residue fields is the identity. For A in \mathcal{C} , consider deformations of $\bar{\rho}$ to $\mathrm{GL}_n(A)$, i.e., continuous homomorphisms $G \rightarrow \mathrm{GL}_n(A)$ that reduce to $\bar{\rho}$ modulo the maximal ideal of A up to conjugation by matrices that reduce to the identity. By the standard theory (see [M]), using that G is finitely generated topologically, there is a versal such deformation $\rho_{\mathcal{R}} : G_F \rightarrow \mathrm{GL}_n(\mathcal{R})$ with $\mathcal{R} \in \mathcal{C}$.

We need a lemma:

Lemma 4.2 *There is continuous map $\pi : \mathcal{R} \rightarrow \mathcal{O}$ of local rings such that the representation $\pi \circ \rho_{\mathcal{R}}$ has the same traces as ρ , and in fact is isomorphic to the chosen integral model of ρ . In the case that $\bar{\rho}$ is residually absolutely irreducible, π is the unique map with the property that the representation $\pi \circ \rho_{\mathcal{R}}$ has the same traces as ρ , and then automatically $\pi \circ \rho_{\mathcal{R}}$ is isomorphic to an integral model of ρ that in this case is unique.*

Proof. Consider $\rho \bmod \ell^s$ for any $s > 0$, that is valued in $\mathrm{GL}_n(\mathcal{O}_{L_s}/\ell^s \mathcal{O}_{L_s})$ by what was said above. For each positive integer s consider $A_s = (\mathcal{O}_{L_1} + \ell \mathcal{O}_{L_2} + \cdots + \ell^{s-1} \mathcal{O}_{L_s})/\ell^s \mathcal{O}_{L_s}$. Then it is easy to see that A_s is an object of \mathcal{C} and $\rho \bmod \ell^s$ is valued in $\mathrm{GL}_n(A_s)$, via the natural inclusion $A_s \hookrightarrow \mathcal{O}/\ell^s \mathcal{O}$. It is also easy to see by inspection that if we reduce A_s modulo its ideal $\ell^{s-1} \mathcal{O}_{L_s}$ we get A_{s-1} . By the versal property of \mathcal{R} , we have a morphism $\pi_s : \mathcal{R} \rightarrow A_s$ such that $\pi_s \circ \rho_{\mathcal{R}}$ is isomorphic to (the chosen integral model of) $\rho \bmod \ell^s$. Here we are using the fact that the image of $\rho \bmod \ell^s$ is valued in the ring that is the image of the composition of inclusions $A_s \hookrightarrow \mathcal{O}_{L_s}/\ell^s \mathcal{O}_{L_s} \hookrightarrow \mathcal{O}/\ell^s$ for all $s > 0$. In particular the traces of $\pi_s \circ \rho_{\mathcal{R}}$ and $\rho \bmod \ell^s$ coincide. At this point, in the case when \mathcal{R} is a universal object (as is the case when $\bar{\rho}$ is

centralised only by scalars) we directly see that the homomorphisms π_s form a compatible sequence which gives a homomorphism $\pi : \mathcal{R} \rightarrow \mathcal{O}$ such that $\pi \circ \rho_{\mathcal{R}}$ is isomorphic to the chosen integral model of ρ (and in particular has the same traces as ρ). In the case when $\bar{\rho}$ is residually absolutely irreducible we get the additional claim of the lemma by the results of [Ca].

In the case when \mathcal{R} is only versal the lemma is slightly more delicate. We need to observe that as the rings A_s have finite cardinality and as \mathcal{R} is topologically finitely generated, for each s there are only finitely many morphisms $\alpha_s : \mathcal{R} \rightarrow A_s$ —in particular, only finitely many with the further property that $\alpha_s \circ \rho_{\mathcal{R}}$ is isomorphic to $\rho \bmod \ell^s$. By a standard compactness argument, there exists a compatible subsequence $\alpha_{t_1}, \alpha_{t_2}, \dots$ of homomorphisms, which gives a homomorphism $\pi : \mathcal{R} \rightarrow \mathcal{O}$ as before. \square

The image \mathcal{S} of \mathcal{R} in \mathcal{O} is a quotient of a complete Noetherian $W(k)$ -algebra and is therefore a complete Noetherian $W(k)$ -algebra itself. We have the following key lemma whose proof we owe to Shankar Sen:

Lemma 4.3 *Let K denote the field of fractions of $W(k)$. Then there exists a finite subextension L of $K \subset \mathbb{C}_\ell$ such that the integral closure of $W(k)$ in $\mathcal{S} \subset \mathbb{C}_\ell$ is contained in L .*

Proof. As $\mathcal{S} \subset \mathbb{C}_\ell$, it is a complete Noetherian integral domain, so by Cohen structure theory ([EGA IV, 0_{IV} 19.8.8]), there exists a subring $\mathcal{S}_0 \subset \mathcal{S}$ such that $\mathcal{S}_0 \cong W(k)[[u_1, \dots, u_r]]$, and \mathcal{S} is a local, module-finite \mathcal{S}_0 -algebra. Suppose it can be generated by s elements. If \mathcal{T} denotes the integral closure of $W(k)$ in \mathcal{S} , every element of $t \in \mathcal{T}$ is integral over \mathcal{S}_0 and therefore satisfies a minimal monic polynomial equation with coefficients in \mathcal{S}_0 and degree $\leq s$. This polynomial divides the minimal polynomial of t over $W(k)$ and therefore has coefficients in $\bar{\mathbb{Q}}_\ell \cap W(k)[[u_1, \dots, u_r]] = W(k)$. It follows that t lies in an extension of K of degree $\leq s$; by Krasner's lemma, the compositum of all such extensions is a finite extension of K . \square

We go back to the proof of Theorem 4.1. Let X denote the union of the conjugacy classes of the $x_i \in G$. We know that $\mathrm{tr}(\rho)$ is continuous and maps X into L , X is dense in G , and L is closed in \mathbb{C}_ℓ . It follows that $\mathrm{tr}(\rho)$ maps G into L . If ρ is semisimple, the theory of pseudorepresentations (see [T]) shows that after replacing L by a suitable finite extension, we can conjugate $\rho(G)$ into $\mathrm{GL}_n(L)$. \square

Corollary 4.4 *Let F be a number field with Galois group $G_F = \text{Gal}(\bar{F}/F)$, ℓ is a rational prime, and $\rho: G_F \rightarrow \text{GL}_n(\mathbb{C}_\ell)$ a continuous, irreducible Galois representation ramified at a finite set of primes. If there exists a set of primes \wp of F of Dirichlet density 1 such that the trace of Frobenius $\text{tr}(\rho(\sigma_\wp))$ belongs to $\mathbb{Q}_\ell \subset \mathbb{C}_\ell$, then ρ is conjugate to a representation with values in a finite extension of \mathbb{Q}_ℓ .*

Proof. This follows from the above theorem, using the following consequence of the Hermite-Minkowski theorem: the Galois group of a pro- ℓ extension of a number field K , ramified only over a finite set of primes S , is topologically finitely generated. \square

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